BE/APh 161: Physical Biology of the Cell, Winter 2014
Short primer on linear stability analysis

Linear stability analysis is a convenient tool to assess the stability of a fixed point of a system of ODEs. Consider such a system,

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = f(\mathbf{u}), \tag{1}$$

where $\mathbf{u}$ is a vector of variables, with $f(\mathbf{u})$ being a vector-valued function. Let $\mathbf{u}_0$ be a fixed point of the system. I.e.,

$$f(\mathbf{u}_0) = 0. \tag{2}$$

No, let us perform a Taylor expansion of $f(\mathbf{u})$ about the fixed point $\mathbf{u} = \mathbf{u}_0$. We assume $f(\mathbf{u})$ is smooth around $\mathbf{u}_0$.

$$f(\mathbf{u}) = f(\mathbf{u}_0) + \nabla f(\mathbf{u}_0) \cdot \delta\mathbf{u} + \text{higher order terms} \tag{3}$$

where $\delta\mathbf{u} \equiv \mathbf{u} - \mathbf{u}_0$. Note that $\nabla f(\mathbf{u}_0)$ is the vector gradient of $f(\mathbf{u})$ evaluated at $\mathbf{u}_0$, since $f(\mathbf{u})$ is a vector-valued function. This means $\nabla f(\mathbf{u}_0)$ is a matrix of derivatives. We will call this matrix A.

$$\mathsf{A} \equiv \nabla f(\mathbf{u}_0), \tag{4}$$

$$A_{ij} = \frac{\partial f_i}{\partial u_j}. \tag{5}$$

Inserting this back into our original set of ODEs (1) and keeping only terms to linear order in the perturbation $\delta\mathbf{u}$ yields

$$\frac{\mathrm{d}\mathbf{u}_0}{\mathrm{d}t} + \frac{\mathrm{d}\delta\mathbf{u}}{\mathrm{d}t} = f(\mathbf{u}_0) + \mathsf{A} \cdot \delta\mathbf{u}. \tag{6}$$

Using the fact that

$$\frac{\mathrm{d}\mathbf{u}_0}{\mathrm{d}t} = f(\mathbf{u}_0) = 0 \tag{7}$$

by the definition of a fixed point, we have

$$\frac{\mathrm{d}\delta\mathbf{u}}{\mathrm{d}t} = \mathsf{A} \cdot \delta\mathbf{u}, \tag{8}$$

which is a linear system of ODEs. This process of taking the Taylor expansion of $f(\mathbf{u})$ to linear order to get a system of linear ODEs is called *linearization*.

If the real part of all of the eigenvalues of A are negative, any perturbation from the fixed point will relax back to the fixed point. If this is the case, the fixed point is stable. If, however, one or more of the eigenvalues of A has a positive real part, any perturbation away from the fixed point will grow exponentially away from the fixed point. When this is the case, the fixed point is unstable. To summarize the stability of a fixed point based on the eigenvalues λ of $\mathbf{A}$,

$$\text{the fixed point } \mathbf{u}_0 \text{ is } \begin{cases} \text{stable} & \text{if } \mathrm{Re}[\lambda] < 0 \text{ for all } \lambda \\ \text{unstable} & \text{if } \mathrm{Re}[\lambda] > 0 \text{ for any } \lambda \end{cases} \tag{9}$$

If the fixed point is permanent (meaning it is not created or destroyed), and changing a parameter moves the system from being stable to being unstable, a transcritical bifurcation occurs as the relevant eigenvalues of A have a zero real part.

It is also worth noting that the fastest growing mode of an instability is that with the eigenvalue with the largest real part. If the eigenvalue of the fastest growing mode also has an imaginary part, the instability is oscillatory.

Example: Genetic switch from lecture. In lecture, we considered a genetic switch. The dimensionless ODEs we wrote down were

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -u + \frac{\alpha}{(1+v)^2} \tag{10}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -v + \frac{\alpha}{(1+u)^2}, \tag{11}$$

and we determined that $u = v = \alpha/(1+v)^2$ is a fixed point. We will call this fixed point (u_0, v_0) and assess its stability. To be explicit in the calculation of A, we have

$$A_{11} = \frac{\partial}{\partial u}\left(-u + \frac{\alpha}{(1+v)^2}\right)\bigg|_{u,v=u_0,v_0} = -1 \tag{12}$$

$$A_{12} = \frac{\partial}{\partial v}\left(-u + \frac{\alpha}{(1+v)^2}\right)\bigg|_{u,v=u_0,v_0} = -\frac{2\alpha}{(1+v_0)^3} \tag{13}$$

$$A_{21} = \frac{\partial}{\partial u}\left(-v + \frac{\alpha}{(1+u)^2}\right)\bigg|_{u,v=u_0,v_0} = -\frac{2\alpha}{(1+u_0)^3} \tag{14}$$

$$A_{11} = \frac{\partial}{\partial v}\left(-v + \frac{\alpha}{(1+u)^2}\right)\bigg|_{u,v=u_0,v_0} = -1 \tag{15}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta u\\ \delta v\end{pmatrix} = \begin{pmatrix}-1 & -\frac{2\alpha}{(1+u_0)^3}\\ -\frac{2\alpha}{(1+u_0)^3} & -1\end{pmatrix}\cdot\begin{pmatrix}\delta u\\ \delta v\end{pmatrix}, \tag{16}$$

where we have used the fact that $u_0 = v_0$. The eigenvalues of A are

$$\lambda = -1 \pm \frac{2\alpha}{(1+u_0)^3} = -1 \pm 2\,\frac{u_0}{1+u_0}. \tag{17}$$

Thus, the fixed point is stable for $\alpha = u_0(1+u_0)^2$ such that

$$\frac{u_0}{1+u_0} < \frac{1}{2}, \tag{18}$$

and unstable otherwise. The eigenvalues have no imaginary parts, so the instability will not be oscillatory. A transcritical bifurcation occurs when

$$\frac{u_0}{1+u_0} = \frac{1}{2}, \tag{19}$$

or $u_0 = 1$, which gives $\alpha = u_0(1+u_0)^2 = 4$.