BE/APh 161: Physical Biology of the Cell, Winter 2014
Handout: storage and loss moduli of a Maxwell material

In this handout, we show the calculation for the storage and loss moduli of a Maxwell material. Recall that the constitutive relation for a Maxwell material is

$$\sigma + \tau_M \dot{\sigma} = \eta \dot{\epsilon}, \tag{1}$$

where $\tau_M = \eta/E$ is the Maxwell time. When we impose a stress of

$$\sigma(t) = \sigma_0 \sin \omega t, \tag{2}$$

we measure the strain $\epsilon(t)$ as

$$\epsilon(t) = \epsilon_0 \sin(\omega t - \delta), \tag{3}$$

where ϵ_0 is the amplitude of the strain and δ is the phase shift between the applied stress and the resultant strain. The storage and loss moduli are respectively defined as

$$E' \equiv \frac{\sigma_0}{\epsilon_0} \cos \delta \tag{4}$$

$$E'' \equiv \frac{\sigma_0}{\epsilon_0} \sin \delta. \tag{5}$$

These are sometimes also known as the elastic and viscous moduli, respectively, and are sometimes denoted G' and G''.

To compute E' and E'' as a function of E and η, we insert the imposed stress (2) into the constitutive relation (1). The result is

$$\sigma_0(\sin \omega t + \tau_M \omega \cos \omega t) = \eta \dot{\epsilon}. \tag{6}$$

We rearrange this to get

$$\dot{\epsilon} = \frac{\sigma_0}{\eta}(\sin \omega t + \tau_M \omega \cos \omega t). \tag{7}$$

This equation is integrated to give

$$\epsilon(t) = \sigma_0 \left(-\frac{\cos \omega t}{\eta \omega} + \frac{\sin \omega t}{E} \right) + C, \tag{8}$$

were C is a constant of integration. Since $\epsilon = 0$ when $\sigma_0 = 0$ (we have no strain in the absence of stress), $C = 0$, giving

$$\epsilon(t) = \sigma_0 \left(-\frac{\cos \omega t}{\eta \omega} + \frac{\sin \omega t}{E} \right). \tag{9}$$

As an aside, we can rewrite (9) as

$$E\epsilon(t) = \sigma_0 \left(-\frac{\cos \omega t}{\tau_M \omega} + \sin \omega t \right). \tag{10}$$

Note that if $\tau_M \gg \omega^{-1}$, the first term in the parentheses is tiny compared to the second, and

$$E\epsilon \approx \sigma_0 \sin \omega t = \sigma(t), \tag{11}$$

the constitutive relation for a purely elastic solid. If $\tau_M \ll \omega^{-1}$, the second term is tine compared to the first, and

$$E\epsilon \approx -\frac{\sigma_0}{\tau_M \omega} \cos \omega t \quad \Rightarrow \quad \dot{\epsilon} = \frac{\sigma_0}{\eta} \sin \omega t = \frac{\sigma}{\eta}, \tag{12}$$

the constitutive relation for a purely viscous fluid. So, we indeed have the expected limiting behavior and high and low frequencies.

Now, we can rewrite (9) using the trigonometric identity

$$a \cos x + b \sin x = \sqrt{a^2 + b^2} \, \sin(x + \delta), \tag{13}$$

with

$$\delta = \tan^{-1} \frac{a}{b}. \tag{14}$$

We get

$$\epsilon(t) = \left(\frac{\sigma_0^2}{(\eta\omega)^2} + \frac{\sigma_0^2}{E^2} \right)^{\frac{1}{2}} \sin(\omega t - \delta), \tag{15}$$

with

$$\tan \delta = \frac{E}{\eta\omega} = \frac{1}{\tau_M \omega}. \tag{16}$$

From these expressions, we identify the strain amplitude as

$$\epsilon_0 = \sigma_0 \left(\frac{1}{(\eta\omega)^2} + \frac{1}{E^2} \right)^{\frac{1}{2}} = \frac{\sigma_0}{E} \left(1 + \left(\frac{E}{\eta\omega} \right)^2 \right)^{\frac{1}{2}} = \frac{\sigma_0}{E} \left(1 + \tan^2 \delta \right)^{\frac{1}{2}}. \tag{17}$$

We now make use of another trigonometric identity,

$$\tan x = \pm\sqrt{\sec^2 x - 1}. \tag{18}$$

Using this identity, we get

$$\epsilon_0 = \frac{\sigma_0}{E} \left(\sec^2 \delta \right)^{\frac{1}{2}} = \frac{\sigma_0}{E \cos \delta}. \tag{19}$$

From this expression, we have

$$\cos \delta = \frac{\sigma_0}{E\epsilon_0}. \tag{20}$$

Thus,

$$E' \equiv \frac{\sigma_0}{\epsilon_0} \cos \delta = \frac{\sigma_0^2}{E\epsilon_0^2}. \tag{21}$$

But from (17), we have

$$\epsilon_0 = \sigma_0 \left(\frac{1}{(\eta\omega)^2} + \frac{1}{E^2} \right)^{\frac{1}{2}}. \tag{22}$$

Inserting this expression into our equation for E' yields

$$E' = \frac{1}{E} \left(\frac{1}{(\eta\omega)^2} + \frac{1}{E^2} \right)^{-1} = \frac{1}{E} \left(\frac{E^2 + (\eta\omega)^2}{E^2 (\eta\omega)^2} \right)^{-1} = \frac{E(\eta\omega)^2}{E^2 + (\eta\omega)^2}. \tag{23}$$

We have thus identified the storage modulus E'.

We take a similar approach to find E''. We revisit (17) and express it in a different way.

$$\epsilon_0 = \sigma_0 \left(\frac{1}{(\eta\omega)^2} + \frac{1}{E^2} \right)^{\frac{1}{2}} = \frac{\sigma_0}{\eta\omega} \left(1 + \left(\frac{\eta\omega}{E} \right)^2 \right)^{\frac{1}{2}} = \frac{\sigma_0}{\eta\omega} \left(1 + \frac{1}{\tan^2 \delta} \right)^{\frac{1}{2}}. \tag{24}$$

We use the trigonometric identity

$$\cot x = \pm\sqrt{\csc^2 x - 1} \tag{25}$$

to get

$$\epsilon_0 = \frac{\sigma_0}{\eta\omega} \left(\csc^2 \delta \right)^{\frac{1}{2}} = \frac{\sigma_0}{\eta\omega \sin\delta} \quad \Rightarrow \quad \sin\delta = \frac{\sigma_0}{\eta\omega\epsilon_0}. \tag{26}$$

We can now proceed to compute the loss modulus.

$$E'' \equiv \frac{\sigma_0}{\epsilon_0} \sin\delta = \frac{\sigma_0^2}{\epsilon_0^2 \eta\omega} = \frac{1}{\eta\omega} \left(\frac{1}{(\eta\omega)^2} + \frac{1}{E^2} \right)^{-1} = \frac{E^2 \eta\omega}{E^2 + (\eta\omega)^2}. \tag{27}$$

So, in summary,

$$E' = \frac{E(\eta\omega)^2}{E^2 + (\eta\omega)^2} = E\, \frac{(\tau_M \omega)^2}{1 + (\tau_M\omega)^2}, \tag{28}$$

$$E'' = \frac{E^2 \eta\omega}{E^2 + (\eta\omega)^2} = E\, \frac{\tau_M \omega}{1 + (\tau_M\omega)^2}. \tag{29}$$

A plot of the storage and loss moduli is shown in Figure 1.

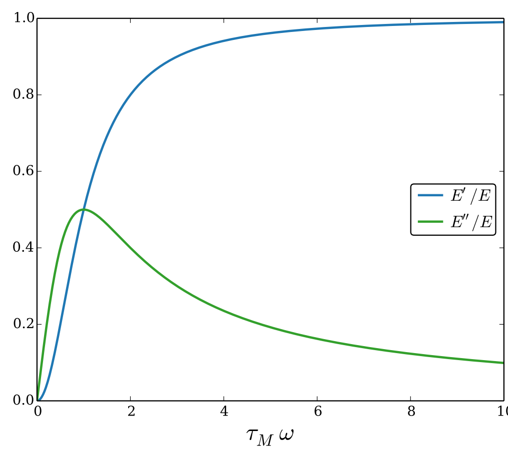

Figure 1: Storage and loss moduli for a Maxwell material. Note that $\lim_{\omega\to\infty} E'' = 0$, though the asymptote is not reached in this plot. Viscous damping is therefore present even at time scales an order of magnitude smaller than the Maxwell time. Notice that the storage and loss moduli are equal when $omega^{-1} = \tau_M$.